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2003 J. Phys. A: Math. Gen. 36 2737

(http://iopscience.iop.org/0305-4470/36/11/306)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 2737–2746 PII: S0305-4470(03)52144-8

Quantum evolution near unstable equilibrium point: an algebraic approach

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Received 1 August 2002
Published 6 March 2003
Online at stacks.iop.org/JPhysA/36/2737

Abstract
We study the quantum evolution of an unstable system in $su(1, 1)$ algebra. The evolution of any initial state $|k, \nu\rangle$ is recursively obtained. When $t \to \infty$, $\left|\langle k', \nu| \exp\left(-\frac{\mathrm{i}}{\hbar} Ht\right)|k, \nu\rangle\right|^2$ decays as $\mathrm{e}^{-4\nu t}$ or $t^{-4\nu}$ in the hyperbolic $(H = 2K_1)$ or parabolic $(H = 2K_1 + 2K_3)$ unstable cases, respectively. The quantum–classic correspondence independent of the Bargmann index ν is established based on the long-time and large-scale behaviour of wavefunctions.

PACS numbers: 02.20.−a, 03.65.Sq, 05.45.Mt

1. Introduction

An equilibrium point is the simplest orbit in classical mechanics. Finding all equilibrium points and analysing their nearby linearized motion often provides us a starting point towards an understanding of the more interesting global dynamics. The linearized motion near a generic equilibrium point is determined by a quadric Hamiltonian. According to the eigenvalue of the resulting linear system, an equilibrium point is characterized as stable (elliptic), unstable (hyperbolic) or marginal stable (parabolic). When turning to quantum mechanics, one can naturally expect that the evolution near a classic equilibrium point is also controlled by the simplified Hamiltonian. If a system is governed by a quadric Hamiltonian, its phase-space (Wigner) representation of density matrix $\mathcal{W}((p, q); t)$ evolves exactly in a classical way, i.e.,

$$\mathcal{W}((q, p); t) = \mathcal{W}(g^{-t}(q, p); 0) \tag{1.1}$$

where g^t is the corresponding classical Hamiltonian flow [1]. Hence the evolution of a quantum state initially localized near a stable equilibrium point is largely, as long as tunnelling is neglected, a localized wavepacket while that near a hyperbolic or parabolic equilibrium point is a transient wave. In this paper, we are interested in the latter case, specifically, the semiclassical understanding of different decay behaviour in an algebraic approach.

We consider the quadric Hamiltonian

$$H(x, p) = \frac{\alpha + 1}{2} x^2 + \frac{\alpha - 1}{2} p^2. \tag{1.2}$$

$(x, p) = (0, 0)$ is a hyperbolic equilibrium point when $|\alpha| < 1$ and a parabolic one when $\alpha = \pm 1$. Let

$$K_1 = \tfrac{1}{4}(x^2 - p^2) \qquad K_2 = -\tfrac{1}{4}(xp + px) \qquad K_3 = \tfrac{1}{4}(x^2 + p^2). \tag{1.3}$$

K_1, K_2, K_3 form a $su(1,1)$ algebra:

$$[K_1, K_2] = -\mathrm{i}\hbar K_3 \qquad [K_2, K_3] = \mathrm{i}\hbar K_1 \qquad [K_3, K_1] = \mathrm{i}\hbar K_2 \tag{1.4}$$

and $H = 2(K_1 + \alpha K_3)$. (This construction has many applications in quantum optics, e.g., [2].) We shall study the quantum evolution in the diagonal representation of K_3, i.e.,

$$\begin{aligned} K_3|n, \nu\rangle &= (n + \nu)\hbar|n, \nu\rangle \\ K_+|n, \nu\rangle &= \sqrt{(n+1)(n+2\nu)}\hbar|n+1, \nu\rangle \\ K_-|n, \nu\rangle &= \sqrt{n(n+2\nu-1)}\hbar|n-1, \nu\rangle \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{1.5}$$

where $K_+ = K_1 + \mathrm{i}K_2$ (or $K_- = K_1 - \mathrm{i}K_2$) is the raising (or lowering) operator and the Bargmann index ν labels the irreducible representation. (The eigenvalue of Casimir operator $K^2 = K_3^2 - K_1^2 - K_2^2$ is $\nu(\nu - 1)\hbar^2$.) For the realization (1.3), $\nu = \frac{1}{4}$ or $\frac{3}{4}$ in the invariant space with even or odd parity, respectively. However, as we shall see, there arises no qualitative difference in the quantum–classic correspondence if we only require $\nu > 0$.

The paper is organized as follows. Sections 2 and 3 discuss the evolution in hyperbolic and parabolic unstable systems, respectively. For simplicity, the index ν in states is dropped and $\hbar$ is set to unity. In section 4 we examine the unstable evolution in a finite system. This is followed by a brief summary.

2. Hyperbolic case

Noting that $U^\dagger(K_1 + \alpha K_3)U = \sqrt{1-\alpha^2}K_1$ for $U = \exp[-\mathrm{i}\tanh^{-1}(\alpha)K_2]$, we restrict ourselves to $H = 2K_1$. Suppose the initial state is $|0\rangle$, then the time evolution is given by Perelomov coherent states [3],

$$\exp(-\mathrm{i}Ht)|0\rangle = \exp(-\mathrm{i}tK_+ + \mathrm{i}tK_-)|0\rangle = \sum_{n=0}^{\infty}\left[\frac{\Gamma(n+2\nu)}{\Gamma(2\nu)n!}\right]^{\frac{1}{2}}\frac{(-\mathrm{i}\tanh t)^n}{\cosh^{2\nu} t}|n\rangle. \tag{2.1}$$

The evolution of any state $|n\rangle$ can be generated from this solution. This is done by introducing a lifting polynomial $P_n(x)$ which satisfies $|n\rangle = P_n(H)|0\rangle$. $P_n(x)$ is determined as follows:

$$P_0(x) = 1 \qquad P_1(x) = \frac{x}{\sqrt{2\nu}} \tag{2.2}$$

and

$$P_{k+1}(x) = \frac{1}{\sqrt{(k+1)(k+2\nu)}}[xP_k(x) - \sqrt{k(k+2\nu-1)}P_{k-1}(x)] \tag{2.3}$$

for $k > 1$. So $P_n(x)$ is a degree-n polynomial of x. Note that the recursion relation (2.3) can also be explained as an eigenequation of H, i.e.,

$$|x) = \sum_{k=0}^{\infty} P_k(x)|k\rangle \tag{2.4}$$

is a formal eigenvector of H with x being the eigenvalue. More properties of the lifting polynomials are presented in appendix A.1. Since $\exp(-\mathrm{i}Ht)H = H\exp(-\mathrm{i}Ht) = \mathrm{i}\frac{\partial}{\partial t}\exp(-\mathrm{i}Ht)$, we have

$$\exp(-\mathrm{i}Ht)|n\rangle = \exp(-\mathrm{i}Ht)P_n(H)|0\rangle = P_n\left(\mathrm{i}\frac{\partial}{\partial t}\right)\exp(-\mathrm{i}Ht)|0\rangle \tag{2.5}$$

and

$$\langle m|\exp(-\mathrm{i}Ht)|n\rangle = P_n\left(\mathrm{i}\frac{\partial}{\partial t}\right)\langle m|\exp(-\mathrm{i}Ht)|0\rangle = P_n\left(\mathrm{i}\frac{\partial}{\partial t}\right)P_m\left(\mathrm{i}\frac{\partial}{\partial t}\right)G_0(t) \tag{2.6}$$

where

$$G_0(t) = \langle 0|\exp(-\mathrm{i}Ht)|0\rangle = \frac{1}{\cosh^{2\nu} t}. \tag{2.7}$$

For example,

$$\begin{aligned}
\langle n|\exp(-\mathrm{i}Ht)|1\rangle &= \left[\frac{\Gamma(n+2\nu)}{\Gamma(1+2\nu)n!}\right]^{\frac{1}{2}}[n-(n+2\nu)\tanh^2 t]\frac{(-\mathrm{i}\tanh t)^{n-1}}{\cosh^{2\nu} t}\\
\langle n|\exp(-\mathrm{i}Ht)|2\rangle &= \left[\frac{2\Gamma(n+2\nu)}{\Gamma(2+2\nu)n!}\right]^{\frac{1}{2}}\left[\frac{n(n-1)}{2}-n(n+2\nu)\tanh^2 t\right.\\
&\quad\left.+\frac{(n+2\nu)(n+2\nu+1)}{2}\tanh^4 t\right]\frac{(-\mathrm{i}\tanh t)^{n-2}}{\cosh^{2\nu} t}
\end{aligned} \tag{2.8}$$

and so on.

Equation (2.6) can be rewritten in the spectral decomposition form. For real ϵ, let $|\epsilon\rangle = \sqrt{A(\epsilon)}|\epsilon)$, where $A(\epsilon)$ is the Fourier transition of $G_0(t)$,

$$A(\epsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_0(t)\exp(\mathrm{i}\epsilon t)\,\mathrm{d}t. \tag{2.9}$$

$|\epsilon\rangle$ is the δ-normalized eigenstate of H (see appendix A.2), by which we have

$$\begin{aligned}
\langle m|\exp(-\mathrm{i}Ht)|n\rangle &= P_n\left(\mathrm{i}\frac{\partial}{\partial t}\right)P_m\left(\mathrm{i}\frac{\partial}{\partial t}\right)\int_{-\infty}^{\infty} A(\epsilon)\exp(-\mathrm{i}\epsilon t)\,\mathrm{d}\epsilon\\
&= \int_{-\infty}^{\infty} P_n(\epsilon)P_m(\epsilon)A(\epsilon)\exp(-\mathrm{i}\epsilon t)\,\mathrm{d}\epsilon\\
&= \int_{-\infty}^{\infty}\langle m|\epsilon\rangle\langle\epsilon|n\rangle\exp(-\mathrm{i}\epsilon t)\,\mathrm{d}\epsilon.
\end{aligned} \tag{2.10}$$

When $t > 0$, equation (2.6) can also be represented by exponentially decaying series. Inserting

$$G_0(t) = 2^{2\nu}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(k+2\nu)}{\Gamma(2\nu)k!}\,\mathrm{e}^{-2(k+\nu)t} \tag{2.11}$$

into equation (2.6), we have

$$\langle m|\exp(-\mathrm{i}Ht)|n\rangle = 2^{2\nu}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(k+2\nu)}{\Gamma(2\nu)k!}P_m(-2(k+\nu)\,\mathrm{i})P_n(-2(k+\nu)\mathrm{i})\,\mathrm{e}^{-2(k+\nu)t}. \tag{2.12}$$

This expansion can normally be written as

$$\exp(-\mathrm{i}Ht)|n\rangle = 2^{2\nu}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(k+2\nu)}{\Gamma(2\nu)k!}P_n(-2(k+\nu)\,\mathrm{i})\,\mathrm{e}^{-2(k+\nu)t}\Psi_k \tag{2.13}$$

where $\Psi_k \equiv |-2(k+\nu)\,\mathrm{i})$ is the 'eigenvector' of H with pure imaginary eigenvalue $-2(k+\nu)\,\mathrm{i}$. It should be pointed out that Ψ_k cannot be normalized and hence is not a physical state. ($P_n(-2(k+\nu)\,\mathrm{i}) \sim n^{k+\nu-\frac{1}{2}}$, when $n \to \infty$, see equation (A1.6).) In the theory of rigged Hilbert space, Ψ_k is explained as an element dual to physical state [4].

In equation (2.12), if we fix n, m and let $t \to \infty$, then

$$|\langle m|\exp(-\mathrm{i}Ht)|n\rangle|^2 = 2^{4\nu}\frac{\Gamma(n+2\nu)\Gamma(m+2\nu)}{m!n!(\Gamma(2\nu))^2}\,\mathrm{e}^{-4\nu t} + O(\mathrm{e}^{-(4\nu+2)t}). \quad (2.14)$$

Here we have used the fact $P_n(-2\nu\,\mathrm{i}) = (-\mathrm{i})^n\left[\frac{\Gamma(n+2\nu)}{\Gamma(2\nu)n!}\right]^{\frac{1}{2}}$. This is an expected result in one aspect: a state has an exponentially decaying probability of staying near an unstable equilibrium point. But in another aspect, it is somewhat confusing: except for the case of $\nu = \frac{1}{4}$, the decay exponent 4ν dose not coincide with the Lyapounov characteristic exponent of the equilibrium point. However, the correct quantum–classic correspondence, which should be ν-independent, can be established as follows.

The simplest way is to consider the expectation value of operator K_i. Since the classical and quantum (Heisenberg) equations of K_i take the same form, $K_1 = \text{const}$ and

$$\begin{bmatrix} K_2(t) \\ K_3(t) \end{bmatrix} = \begin{bmatrix} \cosh 2t & \sinh 2t \\ \sinh 2t & \cosh 2t \end{bmatrix} = \begin{bmatrix} K_2(0) \\ K_3(0) \end{bmatrix} \quad (2.15)$$

can be regarded as the evolution of both the classical and quantum expectation values of K_i.

The next point of view concerns the long-time and large-scale behaviour of the wavefunction. We first consider the evolution of $|0\rangle$ and assume that the major part of $\langle n|\exp(-\mathrm{i}Ht)|0\rangle$, when t is sufficiently large, comes from the region where $n \sim \mathrm{e}^{2t}$. In the limit of $n, t \to \infty$ and $w \equiv 4n\,\mathrm{e}^{-2t}$ keeps fixed, according to equation (2.1),

$$\langle n|\exp(-\mathrm{i}Ht)|0\rangle \approx \left(\frac{\Delta w}{\Delta n}\right)^{\frac{1}{2}} \frac{(-\mathrm{i})^n}{(\Gamma(2\nu))^{\frac{1}{2}}} w^{\nu-\frac{1}{2}}\,\mathrm{e}^{-\frac{w}{2}} \equiv (-\mathrm{i})^n \left(\frac{\Delta w}{\Delta n}\right)^{\frac{1}{2}} \Phi_0(w) \quad (2.16)$$

where $\frac{\Delta w}{\Delta n} = 4\mathrm{e}^{-2t}$. Noting that

$$\sum_{n=0}^{\infty} |\langle n|\exp(-\mathrm{i}Ht)|0\rangle|^2 \approx \int_0^{\infty} |\Phi_0(w)|^2\,\mathrm{d}w = 1 \quad (2.17)$$

we have our assumption justified. Equation (2.16) suggests that the evolution of $|0\rangle$ is globally an exponentially spreading and decaying wave. Especially, if the wavefront is defined by

$$\sum_{n=0}^{N_f(t)} |\langle n|\exp(-\mathrm{i}Ht)|0\rangle|^2 = 1 - \eta \quad (2.18)$$

$\eta \approx 0$, then we have $N_f(t) \sim \mathrm{e}^{2t}$, which classically corresponds to $k_3 = \frac{1}{4}(x^2 + p^2) \sim \mathrm{e}^{2t}$. This picture is valid for any initial state $|k\rangle$ (see appendix A.3).

Finally, we consider the asymptotic behaviour of eigenstate $|\epsilon\rangle$. It can be shown that, apart from an oscillatory coefficient, $|\langle n|\epsilon\rangle|^2 \propto n^{-1}$ when $n \to \infty$. This has a straightforward classical explanation. Heuristically, $|\langle n|\epsilon\rangle|^2$ is proportional to the time for a particle with $H = \epsilon$ passing the interval $K_3 \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right]$. In fact, $K_3 \sim \mathrm{e}^{2t}$ implies $1/\dot{K}_3 \sim (K_3)^{-1}$.

3. Parabolic case

Because $\exp(\mathrm{i}\pi K_3)(K_1 - K_3)\exp(-\mathrm{i}\pi K_3) = -(K_1 + K_3)$, we only consider $H = 2(K_1 + K_3)$. For representation (1.2), $H = x^2$ describes the free motion in momentum space.

The evolution of $|0\rangle$ is given by

$$\exp(-\mathrm{i}Ht)|0\rangle = \sum_{n=0}^{\infty} \left[\frac{\Gamma(n+2\nu)}{\Gamma(2\nu)n!}\right]^{\frac{1}{2}} \frac{(-\mathrm{i}t)^n}{(1+\mathrm{i}t)^{n+2\nu}}|n\rangle \quad (3.1)$$

and the lifting polynomials read as

$$P_n(x) = \left[\frac{\Gamma(2\nu)n!}{\Gamma(n+2\nu)}\right]^{\frac{1}{2}} \sum_{j=0}^{n} \frac{(-1)^{n-j}\Gamma(n+2\nu)}{j!(n-j)!\Gamma(j+2\nu)} x^j = (-1)^n \left[\frac{\Gamma(2\nu)n!}{\Gamma(n+2\nu)}\right]^{\frac{1}{2}} L_n^{(2\nu-1)}(x) \tag{3.2}$$

where $L_n^{(\mu)}(x)$ is the associated Laguerre polynomial. When $t \to \infty$,

$$G_0(t) = \langle 0|\exp(-\mathrm{i}Ht)|0\rangle = (1+\mathrm{i}t)^{-2\nu} = \mathrm{e}^{-\mathrm{i}\nu\pi} t^{-2\nu} + O(t^{-(2\nu+1)}) \tag{3.3}$$

which implies

$$|\langle m|\exp(-\mathrm{i}Ht)|n\rangle|^2 = \frac{\Gamma(n+2\nu)\Gamma(m+2\nu)}{m!n!(\Gamma(2\nu))^2} t^{-4\nu} + O(t^{-(4\nu+1)}). \tag{3.4}$$

Therefore, the decay of a state near a parabolic equilibrium point follows a power law and the power exponent relies on ν. As is demonstrated in the hyperbolic case, ν-independent quantum–classic correspondence can be obtained from different points of view. For brevity, we focus on the long-time and large-scale structure of the wavefunction.

We first consider the evolution of $|0\rangle$ and assume that the major contribution of $\langle n|\exp(-\mathrm{i}Ht)|0\rangle$ comes from the region where $n \sim t^2$. Let $w \equiv n/t^2$, when $t \gg 1$,

$$\frac{(\mathrm{i}t)^n}{(1+\mathrm{i}t)^n} = \mathrm{e}^{\mathrm{i}n\theta}\left(1+\frac{1}{t^2}\right)^{-\frac{n}{2}} \approx \mathrm{e}^{\mathrm{i}n\theta - \frac{w}{2}} \tag{3.5}$$

and

$$\langle n|\exp(-\mathrm{i}Ht)|0\rangle \approx (-1)^n \mathrm{e}^{\mathrm{i}(n\theta-\nu\pi)} \left(\frac{\Delta w}{\Delta t}\right)^{\frac{1}{2}} (\Gamma(2\nu))^{-\frac{1}{2}} w^{\nu-\frac{1}{2}} \mathrm{e}^{-\frac{w}{2}} \tag{3.6}$$

where $\theta = \tan^{-1}\frac{1}{t}$ and $\frac{\Delta w}{\Delta t} = \frac{1}{t^2}$. By applying $P_k\left(\mathrm{i}\frac{\partial}{\partial t}\right)$ to both sides of this relation, the long-time and large-scale behaviour of the evolution of $|k\rangle$ can be obtained. Noting that $\mathrm{i}\frac{\partial}{\partial t}\mathrm{e}^{\mathrm{i}n\theta} \approx w\,\mathrm{e}^{\mathrm{i}n\theta}$ and both $\frac{\partial w}{\partial t}$ and $\frac{\partial}{\partial t}\frac{\Delta w}{\Delta t}$ are the higher order smallness, we have

$$\langle n|\exp(-\mathrm{i}Ht)|k\rangle \approx (-1)^{n+k} \mathrm{e}^{\mathrm{i}(n\theta-\nu\pi)} \left(\frac{\Delta w}{\Delta t}\right)^{\frac{1}{2}} \left[\frac{k!}{\Gamma(k+2\nu)}\right]^{\frac{1}{2}} w^{\nu-\frac{1}{2}} L_k^{(2\nu-1)}(w)\,\mathrm{e}^{-\frac{w}{2}}. \tag{3.7}$$

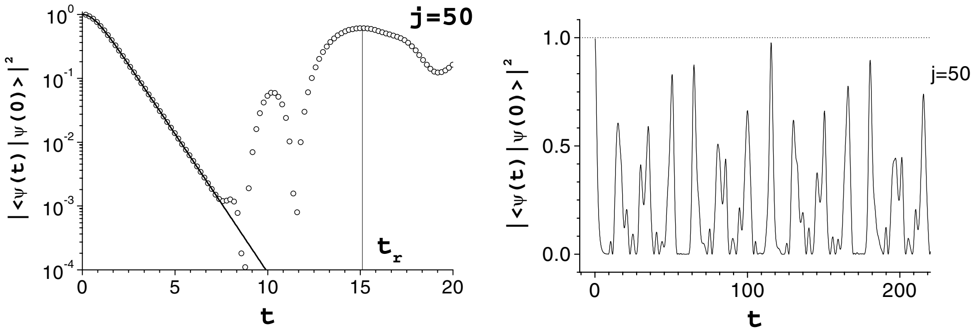
This result implies that the spreading of the state initiated from any $|k\rangle$ and, hence, the superposition of them obeys a power law, $n \propto t^2$.

4. Anisotropic rotator

We have discussed the quantum evolution in linear unstable systems. Different decay behaviour is obtained in the hyperbolic and parabolic cases. Just as in classical mechanics, due to the limitation of the linear approximation and, for a finite system, the revival of the initial state ($\overline{\lim}_{t\to\infty}|\langle\Psi(t)|\Psi(0)\rangle|^2 = 1$), these decay laws describe only local and hence finite time dynamics in a general unstable system. One of the simplest examples of these systems is the anisotropic rotator, which is defined by Hamiltonian

$$H_\alpha(\mathbf{J}) = \tfrac{1}{2}\left(J_1^2 - J_2^2 + \alpha J_3^2\right) \tag{4.1}$$

where J_i is the ith component of angular momentum, $i = 1, 2, 3$. Since $J = \sqrt{J_1^2 + J^2 + J^3}$ is conserved in both classical and quantum mechanics, system (4.1) can be reduced to a classical Hamiltonian system on a sphere with $J = \text{const}$ or a quantum system in each $(2j+1)$-dimensional invariant subspace with $J^2 = j(j+1)\hbar^2$. In quantum–classic correspondence, we assume $J = \left(j+\frac{1}{2}\right)\hbar = 1$ so that the semiclassical limit is represented by $j \to \infty$ or



(*a*) (*b*)

Figure 1. (*a*) $\mathcal{P}_0(t)$ (open dots) at short time. The solid line $(1/\cosh(t))$ shows the result of the linear approximation. (*b*) $\mathcal{P}_0(t)$ at large timescale.

$\hbar = 1/\left(j+\frac{1}{2}\right) \to 0$. Denote the eigenvectors of J_3 by $|m\rangle$, $J_3|m\rangle = m\hbar|m\rangle$. Of these $2j+1$ states, $|j\rangle$ is closest to the equilibrium point $(J_1, J_2, J_3) = (0, 0, 1)$, which is hyperbolic stable when $|\alpha| < 1$ or parabolic when $\alpha = \pm 1$. In fact, the stability of the equilibrium point has a clear quantum correspondence, i.e., the restriction of H_α in the invariant space spanned by $|j\rangle$, $|j-2\rangle \ldots$ (or $|j-1\rangle, |j-3\ldots\rangle$) can be approximated by $2(K_1 - \alpha K_3) + \alpha I$ with $\nu = \frac{1}{4}$ (or $\frac{3}{4}$) when $m \approx j$.

For the hyperbolic case, taking $\alpha = 0$ as an example, $|\langle j|\exp\left(-\frac{\mathrm{i}}{\hbar}Ht\right)|j\rangle|^2 \equiv \mathcal{P}_0(t)$ is numerically calculated. The result shows that $\mathcal{P}_0(t)$ decays in accordance with the prediction of the linear approximation when t is small (see figure 1(*a*)). However, when t is large, $\mathcal{P}_0(t)$ exhibits an oscillation with large variance in the waveshape (see figure 1(*b*)). If $|j-1\rangle$ is set as the initial state, a similar result can also be obtained. For convenience, the position of the first prominent peak of $\mathcal{P}_0(t)$, denoted by t_r, is used to mark the first recursion of the initial wavepacket (see figure 1(*a*)). Our numerical result shows that $t_r \approx c + 2\log(j)$ when $j \to \infty$. This has a simple classical explanation. In classic mechanics, all orbits at $H \neq 0$ are periodic ones. (The orbits at $H = 0$ consist of the equilibrium points $(0, 0, 1)$ and $(0, 0, -1)$ and the four heteroclinic orbits (separatrixes) that connect them.) When $H \to 0$, the period diverges as $-2\log|H|$. Semiclassically, $|j\rangle$ corresponds to a distribution centred at $(0, 0, 1)$. The width of the distribution is of the order of $\hbar^{\frac{1}{2}}$ (or $j^{-\frac{1}{2}}$). Therefore, the classical average of $|H|$ vanishes as j^{-1} and hence the period diverges as $2\log(j)$ in the semiclassical limit.

For the parabolic case ($\alpha = 0$), simple calculation gives

$$G_0(t)| = \langle j|\exp\left(-\frac{\mathrm{i}}{\hbar}Ht\right)|j\rangle = \frac{(2j)!}{2^{2j}}\sum_{m=-j}^{j}\frac{\exp(-\mathrm{i}m^2\hbar t)}{(j-m)!(j+m)!}. \tag{4.2}$$

This sum also appears in diffraction physics and its semiclassical behaviour has been studied by Berry [5]. When $t \sim \hbar^{-1}$, noting that $G_0(t + 2\pi/\hbar) = G_0(t)$, $G_0(t)$ exhibit self-similar structure at large j. When $t < c\hbar^{-\frac{1}{2}}$,

$$G_0(t) \approx \frac{1}{\sqrt{\pi j}}\sum_{-j}^{j}\exp\left[-\left(\frac{1}{j}+\mathrm{i}\hbar t\right)m^2\right] \approx \frac{1}{\sqrt{\pi j}}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{1}{j}+\mathrm{i}\hbar t\right)m^2\right]\mathrm{d}m$$
$$\approx \frac{1}{\sqrt{1+\mathrm{i}t}} \tag{4.3}$$

as the linear approximation implied.

5. Summary

In the above we have discussed the unstable quantum evolution in $su(1,1)$ algebra. For the hyperbolic case ($H = 2K_1$), the evolution of $|0\rangle$ is given by Perelomov coherent states, from which the evolution of an arbitrary state $|k\rangle$ can be recursively generated. When $t \to \infty, |\langle\Psi(t)|\Psi(0)\rangle|^2 \sim \mathrm{e}^{-4\nu t}$, where ν is the Bargmann index. For the parabolic case ($H = 2K_1 + 2K_3$), a close form of the evolution of $|k\rangle$ is also obtained, which shows that $|\langle\Psi(t)|\Psi(0)\rangle|^2 \sim t^{-4\nu}$ at large t. The ν-dependent decay exponents have only quantum meaning. Correct quantum–classic correspondence is established based on the global behaviour of the long-time evolution wavefunctions. Moreover, we demonstrate the limitation of the linear approximation by a simple example, the anisotropic rotator.

Appendix

A.1. Properties of lifting polynomials

Generating function. Equation (2.1) can be rewritten as

$$\exp(-\mathrm{i}Ht)|0\rangle = \sum_{n=0}^{\infty}\left[\frac{\Gamma(n+2\nu)}{\Gamma(2\nu)n!}\right]^{\frac{1}{2}}\frac{(-\mathrm{i}\tanh t)^n}{\cosh^{2\nu} t}P_n(H)|0\rangle \equiv f(H,t)|0\rangle. \tag{A1.1}$$

Applying $P_k(H)$ to both sides of this equation yields $\exp(-\mathrm{i}Ht)|k\rangle = f(H,t)|k\rangle$, which implies $\exp(-\mathrm{i}Ht) = f(H,t)$. Since $I, H, H^2, H^3, \ldots$ are linearly independent, we conclude that $\exp(-\mathrm{i}xt) = f(x,t)$, or, writing $s = -\mathrm{i}\tanh(\mathrm{i}t)$,

$$\frac{\exp(x \arctan s)}{(1+s^2)^\nu} = \sum_{n=0}^{\infty}\left[\frac{\Gamma(n+2\nu)}{\Gamma(2\nu)n!}\right]^{\frac{1}{2}} P_n(x)s^n. \tag{A1.2}$$

This is the generating function of $P_n(x)$.

Asymptotic behaviour of $P_n(-2(k+\nu)\,\mathrm{i})$ at large n. Let $x = -2(k+\nu)\,\mathrm{i}$ in equation (A1.2),

$$\frac{(1-\mathrm{i}s)^k}{(1+\mathrm{i}s)^{k+2\nu}} = \sum_{n=0}^{\infty}\left[\frac{\Gamma(n+2\nu)}{\Gamma(2\nu)n!}\right]^{\frac{1}{2}} P_n(-2(k+\nu)\,\mathrm{i})s^n. \tag{A1.3}$$

Writing $P_n(-2(k+\nu)\,\mathrm{i}) = (-\mathrm{i})^n[\Gamma(n+2\nu)/(\Gamma(2\nu)n!)]^{\frac{1}{2}}t_{n,k}$, equation (A1.3) implies

$$\frac{(1+s)^k}{(1-s)^{k+2\nu}} = \sum_{n=0}^{\infty}\frac{\Gamma(n+2\nu)}{\Gamma(2\nu)n!}t_{n,k}s^n \tag{A1.4}$$

and

$$\sum_{n,k=0}^{\infty}\frac{\Gamma(n+2\nu)\Gamma(k+2\nu)}{\Gamma(2\nu)\Gamma(2\nu)n!k!}t_{n,k}s^n t^k = \frac{1}{(1-s-t-st)^{2\nu}}. \tag{A1.5}$$

So $t_{n,k} = t_{k,n}$, or

$$P_n(-2(k+\nu)\,\mathrm{i}) = \mathrm{i}^{k-n}\left[\frac{k!\Gamma(n+2\nu)}{n!\Gamma(k+2\nu)}\right]^{\frac{1}{2}} P_k(-2(n+\nu)\,\mathrm{i}). \tag{A1.6}$$

Noting that the leading term in $P_k(x)$ is $[\Gamma(2\nu)/(k!\Gamma(k+2\nu))]^{\frac{1}{2}}x^k$, we have

$$P_n(-2(k+\nu)\,\mathrm{i}) = (-\mathrm{i})^n\frac{2^k[\Gamma(2\nu)]^{\frac{1}{2}}}{\Gamma(k+2\nu)}n^{k+\nu-\frac{1}{2}} + O\left(n^{k+\nu-\frac{3}{2}}\right) \tag{A1.6}$$

when $n \to \infty$.

Asymptotic behaviour of $P_n(\epsilon)$ at large n. According to equation (A2.6), for real ϵ, $P_n(\epsilon)$ can be expressed as

$$P_n(\epsilon) = \frac{1}{2\pi A(\epsilon)} \left[\frac{\Gamma(n+2\nu)}{\Gamma(2\nu)n!}\right]^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{(-\mathrm{i}\tanh t)^n}{\cosh^{2\nu} t} \mathrm{e}^{\mathrm{i}\epsilon t}\,\mathrm{d}t. \tag{A1.7}$$

If $n \gg 1$,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{(-\mathrm{i}\tanh t)^n}{\cosh^{2\nu} t} \mathrm{e}^{\mathrm{i}\epsilon t}\,\mathrm{d}t &= 2\,\mathrm{Re}\left\{\int_0^{\infty} \frac{(-\mathrm{i}\tanh t)^n}{\cosh^{2\nu} t} \mathrm{e}^{\mathrm{i}\epsilon t}\,\mathrm{d}t\right\} \\
&\approx 2^{2\nu+1}\,\mathrm{Re}\left\{(-\mathrm{i})^n \int_0^{\infty} \exp[-2n\exp(-2t)]\,\mathrm{e}^{(\mathrm{i}\epsilon-2\nu)t}\,\mathrm{d}t\right\} \\
&= \left(\frac{2}{n}\right)^{\nu} \mathrm{Re}\left\{(-\mathrm{i})^n \exp\left[\mathrm{i}\frac{\epsilon}{2}\log(2n)\right] \int_0^{2n} w^{\nu-\mathrm{i}\frac{\epsilon}{2}-1}\,\mathrm{e}^{-w}\,\mathrm{d}w\right\} \qquad (w \equiv 2n\,\mathrm{e}^{-2t}) \\
&\approx \left(\frac{2}{n}\right)^{\nu} \mathrm{Re}\left\{\Gamma\left(\nu - \mathrm{i}\frac{1}{2}\epsilon\right) \exp\left[\mathrm{i}\frac{\epsilon}{2}\log(2n) - \mathrm{i}\frac{n\pi}{2}\right]\right\}.
\end{aligned} \tag{A1.8}$$

So

$$P_n(\epsilon) \approx \frac{2^{\nu} n^{-\frac{1}{2}}}{2\pi A(\epsilon)[\Gamma(2\nu)]^{\frac{1}{2}}} \mathrm{Re}\left\{\Gamma\left(\nu - \mathrm{i}\frac{\epsilon}{2}\right) \exp\left[\mathrm{i}\frac{\epsilon}{2}\log(2n) - \mathrm{i}\frac{n\pi}{2}\right]\right\} \tag{A1.9}$$

when $n \to \infty$.

A.2. Orthonormality of $|\epsilon\rangle$

We first prove

$$A(\epsilon,\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\epsilon t}}{\cosh^{\alpha}(t)} > 0 \tag{A2.1}$$

for arbitrary $\alpha > 0$ and $\epsilon \in \mathbf{R}$. First we note that

$$A(\epsilon, 1) = \frac{1}{2\cosh\left(\frac{\pi}{2}\epsilon\right)} > 0. \tag{A2.2}$$

Then, making use of the fact

$$A(\epsilon, \alpha_1+\alpha_2) = A(\epsilon,\alpha_1) * A(\epsilon,\alpha_2) = \int_{-\infty}^{\infty} A(\epsilon-\epsilon',\alpha_1)A(\epsilon',\alpha_2)\,\mathrm{d}\epsilon' \tag{A2.3}$$

we conclude that $A(\epsilon, \sum_i \alpha_i) > 0$ if each $A(\epsilon,\alpha_i) > 0$. Specifically, equation (A2.2) implies $A(\epsilon, k) > 0$, $k = 1, 2, \ldots$. Moreover,

$$\frac{1}{\cosh^{\alpha} t} = \frac{1}{(2\cosh^2\frac{t}{2} - 1)^{\alpha}} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)n!2^{n+\alpha}\left(\cosh\frac{t}{2}\right)^{2(n+\alpha)}} \tag{A2.4}$$

implies

$$A(\epsilon,\alpha) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)n!2^{(n+\alpha-1)}} A(2\epsilon, 2\alpha+2n). \tag{A2.5}$$

In other words, $A(\epsilon, 2\alpha) > 0$ means $A(\epsilon, 2\alpha+2n) > 0$ and hence $A(\epsilon,\alpha) > 0$. Following this line we can prove the inequality (A2.1) for $\alpha \in \mathbf{D_0} = \{n2^{-m}, m, n \in \mathbf{N}\}$. Because $A(\epsilon,\alpha)$ depends smoothly on α and $\mathbf{D_0}$ is dense in $\mathbf{R}^+$, we have $A(\epsilon,\alpha) \geqslant 0$ for all $\alpha > 0$. Obviously, $A(0, 2\alpha) > 0$ for arbitrary $\alpha > 0$. Therefore, $A(\epsilon, 2\alpha+2) = A(\epsilon, 2\alpha) * A(\epsilon, 2) > 0$ and, from equation (A2.5), $A(\epsilon,\alpha) > 0$.

Then we give an integral expression of $|\epsilon\rangle$,

$$|\epsilon\rangle = \frac{1}{2\pi\sqrt{A(\epsilon)}}\int_{-\infty}^{\infty} \exp[\mathrm{i}(\epsilon - H)t]|0\rangle\,\mathrm{d}t. \tag{A2.6}$$

This is true because

$$\begin{aligned}\langle n|\epsilon\rangle &= \frac{1}{2\pi\sqrt{A(\epsilon)}}\int_{-\infty}^{\infty} \exp(\mathrm{i}\epsilon t)\langle n|\exp(-\mathrm{i}Ht)|0\rangle\,\mathrm{d}t\\ &= \frac{1}{2\pi\sqrt{A(\epsilon)}}\int_{-\infty}^{\infty} \exp(\mathrm{i}\epsilon t)P_n\left(\mathrm{i}\frac{\partial}{\partial t}\right)G_0(t)\,\mathrm{d}t\\ &= \frac{1}{2\pi\sqrt{A(\epsilon)}}\int_{-\infty}^{\infty} G_0(t)P_n\left(-\mathrm{i}\frac{\partial}{\partial t}\right)\exp(\mathrm{i}\epsilon t)\,\mathrm{d}t\\ &= \frac{P_n(\epsilon)}{2\pi\sqrt{A(\epsilon)}}\int_{-\infty}^{\infty} G_0(t)\exp(\mathrm{i}\epsilon t)\,\mathrm{d}t\\ &= P_n(\epsilon)\sqrt{A(\epsilon)}.\end{aligned} \tag{A2.7}$$

Therefore,

$$\begin{aligned}\langle\epsilon'|\epsilon\rangle &= \frac{1}{4\pi^2\sqrt{A(\epsilon)A(\epsilon')}}\int\int_{-\infty}^{\infty} \exp[\mathrm{i}(\epsilon t - \epsilon' t')]\langle 0|\exp[-\mathrm{i}H(t-t')]|0\rangle\,\mathrm{d}t'\,\mathrm{d}t\\ &= \frac{1}{4\pi^2\sqrt{A(\epsilon)A(\epsilon')}}\int\int_{-\infty}^{\infty} \exp[\mathrm{i}\epsilon(t-t') + \mathrm{i}(\epsilon-\epsilon')t']G_0(t-t')\,\mathrm{d}t'\,\mathrm{d}t\\ &= \frac{1}{2\pi}\sqrt{\frac{A(\epsilon)}{A(\epsilon')}}\int_{-\infty}^{\infty} \exp[\mathrm{i}(\epsilon-\epsilon')t']\,\mathrm{d}t' = \delta(\epsilon-\epsilon').\end{aligned} \tag{A2.8}$$

Moreover, taking $t = 0$ in equation (2.10), we have

$$\int_{-\infty}^{\infty} \langle m|\epsilon\rangle\langle\epsilon|n\rangle\,\mathrm{d}\epsilon = \delta_{m,n} \tag{A2.9}$$

i.e., the completeness of $\{|\epsilon\rangle\}_{\epsilon\in\mathcal{R}}$.

A.3. Scaling of wavefunctions in long-time evolution

Let $F_n = \left(\frac{\Delta w}{\Delta n}\right)^{\frac{1}{2}} w^{\nu-\frac{1}{2}}\mathrm{e}^{-\frac{w}{2}}$, then

$$\frac{\partial}{\partial t}F_n = -2w\frac{\partial}{\partial w}F_n = (w - 2\nu)F_n. \tag{A3.1}$$

Writing $\langle n|\exp(-\mathrm{i}Ht)|k\rangle \approx (-\mathrm{i})^{n+k}G_k(w)F_n(w)$, we have $G_0 = 1/\sqrt{\Gamma(2\nu)}, G_1 = (2\nu - w)/\sqrt{\Gamma(1+2\nu)}$ and, from the recursion relation of $P_n(x)$,

$$\begin{aligned}G_{k+1} = \frac{(\mathrm{i})^{k+1}}{F_n}P_{k+1}\left(\mathrm{i}\frac{\partial}{\partial t}\right)G_0F_n = \frac{1}{\sqrt{(k+1)(k+2\nu)}}\Bigg[\left(2\nu - w + 2w\frac{\partial}{\partial w}\right)G_k(w)\\ + \sqrt{k(k+2\nu-1)}G_{k-1}(w)\Bigg]\end{aligned} \tag{A3.2}$$

for $k > 1$. It can be easily verified that

$$G_k = \left[\frac{k!}{\Gamma(k+2\nu)}\right]^{\frac{1}{2}} L_k^{(2\nu-1)}(w)$$

where $L_n^{(\mu)}(x)$ is the associated Laguerre polynomial. Therefore,

$$\langle n|\exp(-\mathrm{i}Ht)|k\rangle \approx (-\mathrm{i})^{n+k}\left(\frac{\Delta w}{\Delta n}\right)^{\frac{1}{2}}\left[\frac{k!}{\Gamma(k+2\nu)}\right]^{\frac{1}{2}} w^{\nu-\frac{1}{2}}L_k^{(2\nu-1)}(w)\,\mathrm{e}^{-\frac{w}{2}}$$
$$\equiv (-\mathrm{i})^n\left(\frac{\Delta w}{\Delta n}\right)^{\frac{1}{2}}\Phi_k(w). \qquad \text{(A3.3)}$$

Note that the limiting wave preserves the orthonormality of $\exp(-\mathrm{i}Ht)|k\rangle$, i.e.,

$$\int_0^\infty \Phi_{k'}^*(w)\Phi_k(w)\,\mathrm{d}w = \delta_{k',k}. \qquad \text{(A3.4)}$$

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